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Analytical solution to transient heat conduction in polar coordinates with multiple layers in radial direction

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Abstract

Closed form analytical double-series solution is presented for the multi-dimensional unsteady heat conduction problem in polar coordinates (2-D cylindrical) with multiple layers in the radial direction. Spatially non-uniform, but time-independent, volumetric heat sources are assumed in each layer. Separation of variables method is used to obtain transient temperature distribution. In contrast to Cartesian or cylindrical (r, z) coordinates, eigenvalues in the direction perpendicular to the layers do not explicitly depend on those in the other direction. The implication of the above statement is that the imaginary eigenvalues are precluded from the solution of the problem. However, radial (transverse) eigenvalues are implicitly dependent on the angular eigenvalues through the order of the Bessel functions which constitute the radial eigenfunctions. Therefore, for each eigenvalue in the angular direction, corresponding radial eigenvalues must be obtained. Solution is valid for any combination of homogenous boundary condition of the first or second kind in the angular direction. However, inhomogeneous boundary conditions of the third kind are applied in the radial direction. Proposed solution is also applicable to multiple layers with zero inner radius. An illustrative example problem for the three-layer semi-circular annular region is solved. Results along with the isotherms are shown graphically and discussed.

Keywords: Transient heat conduction; Multi-layer; Polar coordinates; Analytical solution

1. Introduction

Multi-layer materials have attracted considerable attention in modern engineering applications due to added advantage of combining physical, mechanical and thermal properties of different materials. These layered components find a wide range of applications in various automotive, space, chemical, civil and nuclear industries. Therefore, there exists a need to accurately and efficiently determine the heat flux and temperature distributions inside the multiple layers.

Recent advances in computational resources for symbolic manipulations have created renewed interest among researchers [1–4] in developing exact analytical solutions of problems for which numerical solutions are currently more prevalent. Although multi-layer heat conduction problems have been studied in great detail and various solution methods—including *orthogonal and quasi-orthogonal expansion technique* [5–8], *Laplace transform method* [9–11], *Green's function approach* [12–14], *finite integral transform technique* [15]—are readily available, there is continued need to explore newly developed or recently modified methods to solve multi-layer problems for which exact analytical solutions do not exist. Such solutions can help improve computational efficiency of computer codes that currently rely on numerical techniques to solve such problems.

Salt [16,17] addressed time-dependent heat conduction problem by *orthogonal expansion technique*, in a two-dimensional composite slab (Cartesian geometry) with no internal heat source, subjected to homogenous boundary conditions. Later, Mikhailov and Ozisik [18] solved the 3-D transient conduction problem in a Cartesian non-homogenous finite medium. More recently, Haji-Sheikh and Beck [19] applied *Green's function approach* to develop transient temperature solutions in a 3-D Cartesian two-layer orthotropic medium including the effects of contact resistance. Lu et al. [9–11]

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Nomenclature

a_{imp}, b_{imp} coefficients of Bessel functions in transverse (radial) eigenfunction
$A_{\text{in}}, B_{\text{in}}, C_{\text{in}}$ coefficients in Eq. (2)
$A_{\text{out}}, B_{\text{out}}, C_{\text{out}}$ coefficients in Eq. (3)
$C_{1\text{in}}, C_{2\text{in}}, C_{1\text{out}}, C_{2\text{out}}$ coefficients in Eq. (44) dependent on inner and outer surface boundary conditions
D_{mp} coefficient in general solution (Eq. (40)) dependent on initial condition
E_{mp} coefficient in general solution (Eq. (41))
$f_i(r,\theta)$ initial temperature distribution in the ith layer at $t=0$
$g_i(r,\theta)$ volumetric heat source distribution in the ith layer
h_{out} outer surface heat transfer coefficient
J_{β_m} Bessel function of the first kind of order β_m
k_i thermal conductivity of the ith layer
M number of angular eigenfunctions used in the transient solution
M_{ss} number of angular eigenfunctions used in the steady state solution
N_{rmp} norms for r-direction
$N_{\theta m}$ norms for θ-direction
P number of radial eigenfunctions used in the solution corresponding to each angular eigenvalue
r radial coordinate
r_i outer radius for the ith layer
$R_{imp}(\lambda_{imp} r)$ transverse eigenfunctions for the ith layer
t time
$T_i(r,\theta,t)$ temperature distribution for the ith layer
Y_{β_m} Bessel function of the second kind of order β_m
x, y Cartesian coordinates
$x_{ij},\ y_{ij},\ j=1,2,3,4$ elements for $(2n \times 2n)$ matrix in Eq. (44)

Greek symbols

α_i thermal diffusivity of the ith layer
β_m eigenvalues in the angular direction
$\Delta\lambda$ window size for evaluation of radial eigenvalues
ε error
η_m eigenvalues in the y-direction
θ angular coordinate
$\Theta_m(\beta_m\theta)$ eigenfunctions in the angular direction
λ_{imp} transverse (radial) eigenvalues
ν_{imp} eigenvalues in the x-direction
ϕ angle subtended by the multi-layers
ω_1, ω_2 coefficients in $\Theta_m(\beta_m\theta)$ equation

Subscripts and superscripts

i layer or interface number
ss steady-state
$'$ differentiation

developed a novel method by combining *Laplace transform method* and *Separation of variables method* to solve multi-dimensional transient heat conduction problem in a rectangular and cylindrical multi-layer slab with time-dependent periodic boundary condition. Treatment in the cylindrical coordinates is, however, restricted to the r–z coordinates. *Eigenfunction expansion method* is applied by de Monte [20] to solve the unsteady heat conduction problem in a two-dimensional, two-layer isotropic slab subjected to homogenous boundary conditions.

The brief review of relevant literature is by no means exhaustive. However, a literature survey showed that analytical solution for unsteady temperature distribution in polar coordinates with multiple layers has not been developed yet. A large number of applications in industries, including semi-circular fiber insulated heaters, multi-layer insulation materials, arc-shaped magnets (used in automotives), nuclear fuel rods and cylindrical or part-cylindrical building structures would benefit from an exact solution in multiple layers. This paper presents an analytical double-series solution for transient heat conduction in polar coordinates (2-D cylindrical) for multi-layer domain in the radial direction with spatially non-uniform but time-independent volumetric heat sources. Inhomogeneous boundary conditions of the third kind are applied in the direction perpendicular to the layers. However, only homogenous boundary conditions of the first or second kind are applicable on $\theta =$ constant surfaces [20]. Moreover, though the approach is very general and applicable to complete discs ($\phi = 2\pi$, see Fig. 1), specific solution developed in this paper is only applicable to domains with pie slice geometry ($\phi < 2\pi$).

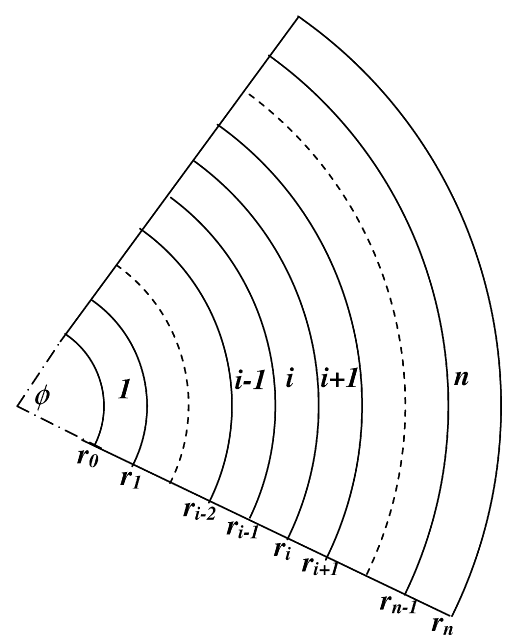


Fig. 1. Schematic representation of n-layers in polar coordinates.

2. Mathematical formulation

Consider an n-layer composite slab in polar coordinates ($r_0 \leqslant r \leqslant r_n$ and $0 \leqslant \theta \leqslant \phi$), as shown schematically in Fig. 1. All the layers are assumed to be isotropic in thermal properties and are in perfect thermal contact. Let k_i and α_i be the temperature independent thermal conductivity and thermal diffusivity of the ith layer. Initially, at $t = 0$, the ith layer is at a specified temperature $f_i(r, \theta)$. For $t > 0$, homogenous boundary conditions of either first or second kind are applied to the angular surfaces at $\theta = 0$ and $\theta = \phi$. All three kinds of boundary conditions are applicable to the inner ($i = 1, r = r_0$) and the outer ($i = n, r = r_n$) radial surfaces. In addition, time independent heat sources $g_i(r, \theta)$ are switched on in each layer at $t = 0$.

Under these assumptions, the governing heat conduction equation, along with the boundary and initial conditions, are as follows:

Governing equation:

$$\frac{1}{\alpha_i}\frac{\partial T_i}{\partial t}(r,\theta,t) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_i}{\partial r}(r,\theta,t)\right) + \frac{1}{r^2}\frac{\partial^2 T_i}{\partial \theta^2}(r,\theta,t) + \frac{g_i(r,\theta)}{k_i}$$
$$r_{i-1} \leqslant r \leqslant r_i,\ 1 \leqslant i \leqslant n \tag{1}$$

Boundary conditions:

- Inner surface of 1st layer ($i = 1$)

$$A_{\text{in}}\frac{\partial T_1}{\partial r}(r_0,\theta,t) + B_{\text{in}}T_1(r_0,\theta,t) = C_{\text{in}} \tag{2}$$

- Outer surface of nth layer ($i = n$)

$$A_{\text{out}}\frac{\partial T_n}{\partial r}(r_n,\theta,t) + B_{\text{out}}T_n(r_n,\theta,t) = C_{\text{out}} \tag{3}$$

- $\theta = 0$ surface ($i = 1, 2, \ldots, n$)

$$T_i(r,\theta=0,t) = 0 \quad \text{or} \quad \frac{\partial T_i}{\partial \theta}(r,\theta=0,t) = 0 \tag{4}$$

- $\theta = \phi$ surface ($i = 1, 2, \ldots, n$)

$$T_i(r,\theta=\phi,t) = 0 \quad \text{or} \quad \frac{\partial T_i}{\partial \theta}(r,\theta=\phi,t) = 0 \tag{5}$$

- Inner interface of ith layer ($i \neq 1$)

$$T_i(r_{i-1},\theta,t) = T_{i-1}(r_{i-1},\theta,t) \tag{6}$$
$$k_i\frac{\partial T_i}{\partial r}(r_{i-1},\theta,t) = k_{i-1}\frac{\partial T_{i-1}}{\partial r}(r_{i-1},\theta,t) \tag{7}$$

- Outer interface of ith layer ($i \neq n$)

$$T_i(r_i,\theta,t) = T_{i+1}(r_i,\theta,t) \tag{8}$$
$$k_i\frac{\partial T_i}{\partial r}(r_i,\theta,t) = k_{i+1}\frac{\partial T_{i+1}}{\partial r}(r_i,\theta,t) \tag{9}$$

Initial condition:

$$T_i(r,\theta,t=0) = f_i(r,\theta) \tag{10}$$

It is to be noted that boundary conditions either of the first, second or third kind can be imposed at $r = r_0$ and $r = r_n$ by choosing the coefficients in Eqs. (2) and (3) appropriately. Furthermore, multiple layers with zero inner radius ($r_0 = 0$) can be simulated by assigning zero values to constants B_{in} and C_{in} in Eq. (2).

3. Solution methodology

In order to apply the *separation of variables method*, which is only applicable to homogenous problems, the non-homogenous problem has to be split into: (1) homogenous transient problem, and (2) non-homogenous steady state problem. This is accomplished by rewriting $T_i(r,\theta,t)$ in the governing equations (1)–(10) as $\overline{T}_i(r,\theta,t) + T_{\text{ss},i}(r,\theta)$, where $\overline{T}_i(r,\theta,t)$ is the "complementary transient" part and $T_{\text{ss},i}(r,\theta)$ is the steady state part of the solution.

3.1. Homogenous transient problem

Homogenized "complementary transient" equations corresponding to Eqs. (1)–(10) are as follows:

Governing equation:

$$\frac{1}{\alpha_i}\frac{\partial \overline{T}_i}{\partial t}(r,\theta,t) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \overline{T}_i}{\partial r}(r,\theta,t)\right) + \frac{1}{r^2}\frac{\partial^2 \overline{T}_i}{\partial \theta^2}(r,\theta,t)$$
$$r_{i-1} \leqslant r \leqslant r_i,\ 1 \leqslant i \leqslant n \tag{11}$$

Boundary conditions:

- Inner surface of 1st layer ($i = 1$)

$$A_{\text{in}}\frac{\partial \overline{T}_1}{\partial r}(r_0,\theta,t) + B_{\text{in}}\overline{T}_1(r_0,\theta,t) = 0 \tag{12}$$

- Outer surface of nth layer ($i = n$)

$$A_{\text{out}}\frac{\partial \overline{T}_n}{\partial r}(r_n,\theta,t) + B_{\text{out}}\overline{T}_n(r_n,\theta,t) = 0 \tag{13}$$

- $\theta = 0$ surface ($i = 1, 2, \ldots, n$)

$$\overline{T}_i(r,\theta=0,t) = 0 \quad \text{or} \quad \frac{\partial \overline{T}_i}{\partial \theta}(r,\theta=0,t) = 0 \tag{14}$$

- $\theta = \phi$ surface ($i = 1, 2, \ldots, n$)

$$\overline{T}_i(r,\theta=\phi,t) = 0 \quad \text{or} \quad \frac{\partial \overline{T}_i}{\partial \theta}(r,\theta=\phi,t) = 0 \tag{15}$$

- Inner interface of ith layer ($i \neq 1$)

$$\overline{T}_i(r_{i-1},\theta,t) = \overline{T}_{i-1}(r_{i-1},\theta,t) \tag{16}$$
$$k_i\frac{\partial \overline{T}_i}{\partial r}(r_{i-1},\theta,t) = k_{i-1}\frac{\partial \overline{T}_{i-1}}{\partial r}(r_{i-1},\theta,t) \tag{17}$$

- Outer interface of ith layer ($i \neq n$)

$$\overline{T}_i(r_i,\theta,t) = \overline{T}_{i-1}(r_i,\theta,t) \tag{18}$$
$$k_i\frac{\partial \overline{T}_i}{\partial r}(r_i,\theta,t) = k_{i-1}\frac{\partial \overline{T}_{i-1}}{\partial r}(r_i,\theta,t) \tag{19}$$

Initial condition:

$$\overline{T}_i(r,\theta,t=0) = f_i(r,\theta) - T_{\text{ss},i}(r,\theta) \tag{20}$$

3.2. Inhomogeneous steady state problem

Inhomogeneous steady state equations corresponding to Eqs. (1)–(10) are as follows:

Governing equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_{\mathrm{ss},i}}{\partial r}(r,\theta)\right)+\frac{1}{r^2}\frac{\partial^2 T_{\mathrm{ss},i}}{\partial\theta^2}(r,\theta)+\frac{g_i(r,\theta)}{k_i}=0$$
$$r_{i-1}\leqslant r\leqslant r_i,\ 1\leqslant i\leqslant n \tag{21}$$

Boundary conditions:

- Inner surface of 1st layer ($i=1$)

$$A_{\mathrm{in}}\frac{\partial T_{\mathrm{ss},1}}{\partial r}(r_0,\theta)+B_{\mathrm{in}}T_{\mathrm{ss},1}(r_0,\theta)=C_{\mathrm{in}} \tag{22}$$

- Outer surface of nth layer ($i=n$)

$$A_{\mathrm{out}}\frac{\partial T_{\mathrm{ss},n}}{\partial r}(r_n,\theta)+B_{\mathrm{out}}T_{\mathrm{ss},n}(r_n,\theta)=C_{\mathrm{out}} \tag{23}$$

- $\theta=0$ surface ($i=1,2,\ldots,n$)

$$T_{\mathrm{ss},i}(r,\theta=0)=0\quad\text{or}\quad\frac{\partial T_{\mathrm{ss},i}}{\partial\theta}(r,\theta=0)=0 \tag{24}$$

- $\theta=\phi$ surface ($i=1,2,\ldots,n$)

$$T_{\mathrm{ss},i}(r,\theta=\phi)=0\quad\text{or}\quad\frac{\partial T_{\mathrm{ss},i}}{\partial\theta}(r,\theta=\phi)=0 \tag{25}$$

- Inner interface of ith layer ($i\neq 1$)

$$T_{\mathrm{ss},i}(r_{i-1},\theta)=T_{\mathrm{ss},i-1}(r_{i-1},\theta) \tag{26}$$
$$k_i\frac{\partial T_{\mathrm{ss},i}}{\partial r}(r_{i-1},\theta)=k_{i-1}\frac{\partial T_{ss,i-1}}{\partial r}(r_{i-1},\theta) \tag{27}$$

- Outer interface of ith layer ($i\neq n$)

$$T_{\mathrm{ss},i}(r_i,\theta)=T_{\mathrm{ss},i+1}(r_i,\theta) \tag{28}$$
$$k_i\frac{\partial T_{\mathrm{ss},i}}{\partial r}(r_i,\theta)=k_{i+1}\frac{\partial T_{\mathrm{ss},i+1}}{\partial r}(r_i,\theta) \tag{29}$$

4. Solution to the homogenous transient problem

4.1. Separation of variables

Substituting the product form for temperature $\overline{T}_i(r,\theta,t)$,

$$\overline{T}_i(r,\theta,t)=R_i(r)\Theta_i(\theta)\Gamma_i(t) \tag{30}$$

in Eq. (11), and then applying separation of variables, yield the following ODEs:

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}r\frac{\mathrm{d}R_i}{\mathrm{d}r}+\left(-\frac{\beta_i^2}{r^2}+\lambda_i^2\right)R_i=0 \tag{31}$$
$$\frac{\mathrm{d}^2\Theta_i}{\mathrm{d}\theta^2}+\beta_i^2\Theta_i=0 \tag{32}$$
$$\frac{1}{\alpha_i}\frac{\mathrm{d}\Gamma_i}{\mathrm{d}t}+\lambda_i^2\Gamma_i=0 \tag{33}$$

where λ_i^2 and β_i^2 are constants of separation.

4.2. General solution

For heat flux to be continuous at the layer interfaces, namely Eqs. (17) and (19), for all values of t [7,16,20,21],

$$\lambda_{imp}=\lambda_{1mp}\sqrt{\alpha_1/\alpha_i},\quad i=1,2,\ldots,n \tag{34}$$

and also

$$\Theta_i=\Theta\quad\Rightarrow\quad\beta_i=\beta,\quad i=1,2,\ldots,n \tag{35}$$

Now, the eigenfunctions $R_{imp}(\lambda_{imp}r)$ corresponding to eigenvalue problem in the r-direction are given by:

$$R_{imp}(\lambda_{imp}r)=a_{imp}J_{\beta_m}(\lambda_{imp}r)+b_{imp}Y_{\beta_m}(\lambda_{imp}r) \tag{36}$$

Orthogonality condition for the r-direction eigenfunctions, which is similar to that in [21], is:

$$\sum_{i=1}^{n}\frac{k_i}{\alpha_i}\int_{r_{i-1}}^{r_i}r R_{imp}(\lambda_{imp}r)R_{imq}(\lambda_{imq}r)\,\mathrm{d}r=\begin{bmatrix}0 & \text{if } p\neq q\\ N_{rmp} & \text{if } p=q\end{bmatrix} \tag{37}$$

Proof of the above condition is given in Appendix A.

Similarly, eigenfunctions $\Theta_m(\beta_m\theta)$ corresponding to the eigenvalue problem in the θ-direction are given by:

$$\Theta_m(\beta_m\theta)=\omega_1\sin(\beta_m\theta)+\omega_2\cos(\beta_m\theta) \tag{38}$$

where constants ω_1, ω_2 and β_m are listed in Table 1 for different combinations of boundary conditions at $\theta=0$ and $\theta=\phi$ edges.

Orthogonality condition for the θ-direction eigenfunctions is:

$$\int_0^{\phi}\Theta_m(\beta_m\theta)\Theta_l(\beta_l\theta)\,\mathrm{d}\theta=\begin{bmatrix}0 & \text{if } m\neq l\\ N_{\theta m} & \text{if } m=l\end{bmatrix} \tag{39}$$

In view of the equations listed before, a general solution for Eq. (11) may be considered as:

$$\overline{T}_i(r,\theta,t)=\sum_{m=1}^{\infty}\sum_{p=1}^{\infty}D_{mp}\mathrm{e}^{-\alpha_i\lambda_{imp}^2t}R_{imp}(\lambda_{imp}r)\Theta_m(\beta_m\theta) \tag{40}$$

It should be noted here that the formulation given above is valid only for polar angle $\phi<2\pi$. For the case of periodic boundary conditions, which is for $\phi=2\pi$, the general solution will be the sum of two double series solutions and may not be directly extracted from the analytical solution obtained in this paper.

Table 1
ω_1, ω_2 and β_m for different combinations of boundary conditions at $\theta=0$ and $\theta=\phi$ surfaces

BC at $\theta=0$	BC at $\theta=\phi$	ω_1	ω_2	β_m
$\overline{T}_i(r,\theta=0,t)=0$	$\overline{T}_i(r,\theta=\phi,t)=0$	1	0	$\frac{m\pi}{\phi}$
$\frac{\partial\overline{T}_i}{\partial\theta}(r,\theta=0,t)=0$	$\frac{\partial\overline{T}_i}{\partial\theta}(r,\theta=\phi,t)=0$	0	1	$\frac{m\pi}{\phi}$
$\overline{T}_i(r,\theta=0,t)=0$	$\frac{\partial\overline{T}_i}{\partial\theta}(r,\theta=\phi,t)=0$	1	0	$\frac{2m-1}{2}\frac{\pi}{\phi}$
$\frac{\partial\overline{T}_i}{\partial\theta}(r,\theta=0,t)=0$	$\overline{T}_i(r,\theta=\phi,t)=0$	0	1	$\frac{2m-1}{2}\frac{\pi}{\phi}$

$$\begin{bmatrix} C_{1\text{in}} & C_{2\text{in}} & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ x_{11} & x_{12} & x_{13} & x_{14} & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ y_{11} & y_{12} & y_{13} & y_{14} & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & x_{i1} & x_{i2} & x_{i3} & x_{i4} & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & y_{i1} & y_{i2} & y_{i3} & y_{i4} & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & x_{n-1,1} & x_{n-1,2} & x_{n-1,3} & x_{n-1,4} \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & y_{n-1,1} & y_{n-1,2} & y_{n-1,3} & y_{n-1,4} \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & C_{1\text{out}} & C_{2\text{out}} \end{bmatrix} \begin{bmatrix} a_{1mp} \\ b_{1mp} \\ \dots \\ \dots \\ a_{imp} \\ b_{imp} \\ \dots \\ \dots \\ a_{nmp} \\ b_{nmp} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dots \\ \dots \\ 0 \\ 0 \\ \dots \\ \dots \\ 0 \\ 0 \end{bmatrix} \tag{44}$$

4.3. Absence of imaginary radial eigenvalues

In general, for multi-layer time-dependent heat conduction problems in Cartesian coordinates, transverse eigenvalues may be imaginary. Same is true for 2-D (r, z) cylindrical coordinates. The eigenvalues are imaginary due to the explicit dependence of the transverse eigenvalues on those in the remaining direction(s). For example, in 2-D Cartesian two-layer (layers in x-direction) homogenous heat conduction problem, general solution in the ith layer is as follows [16,19,20]:

$$T_i(x, y, t) = \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} E_{mp} \mathrm{e}^{-\alpha_i (\nu_{imp}^2 + \eta_m^2) t} X_{imp}(\nu_{imp} x) Y_m(\eta_m y) \tag{41}$$

For heat flux to be continuous at the interface, for all values of t

$$\alpha_1 (\nu_{1mp}^2 + \eta_m^2) = \alpha_2 (\nu_{2mp}^2 + \eta_m^2) \tag{42}$$

which implies,

$$\nu_{2mp} = \sqrt{\frac{\alpha_1}{\alpha_2} \nu_{1mp}^2 + \left(\frac{\alpha_1}{\alpha_2} - 1 \right) \eta_m^2} \tag{43}$$

Clearly, above relation may result in either real or imaginary transverse eigenvalues [16,20].

However, in the present case, as shown in Section 4.2, similar considerations led to Eq. (34), which is similar to what has been established for 1-D multi-layer, time-dependent problems and eliminates the possibility of imaginary eigenvalues. It should be noted that, though there is no explicit dependence between radial and angular eigenvalues, order of the Bessel functions constituting radial eigenfunctions is determined by the angular eigenvalues. Hence, the radial eigenvalues *implicitly* depend on the angular eigenvalues. Moreover, unlike in Cartesian coordinates, this implicit dependence does not vanish even if $\alpha_1 = \alpha_i$ $(i \neq 1)$. In fact, it exists even for single-layer problems.

4.4. Radial eigencondition

Application of the interface conditions (16)–(19) and boundary conditions (12), (13) to the transverse eigenfunction $R_{imp}(\lambda_{imp} r)$ yields, for each integer value of m, the $(2n \times 2n)$ matrix equation (44) shown at the top of this page, where

$$C_{1\text{in}} = A_{\text{in}} J'_{\beta_m}(\lambda_{1mp} r_0) + B_{\text{in}} J_{\beta_m}(\lambda_{1mp} r_0)$$
$$C_{2\text{in}} = A_{\text{in}} Y'_{\beta_m}(\lambda_{1mp} r_0) + B_{\text{in}} Y_{\beta_m}(\lambda_{1mp} r_0)$$
$$x_{i1} = J_{\beta_m}(\lambda_{imp} r_i)$$
$$x_{i2} = Y_{\beta_m}(\lambda_{imp} r_i)$$
$$x_{i3} = -J_{\beta_m}(\lambda_{i+1,mp} r_i)$$
$$x_{i4} = -Y_{\beta_m}(\lambda_{i+1,mp} r_i)$$
$$y_{i1} = k_i J'_{\beta_m}(\lambda_{imp} r_i)$$
$$y_{i2} = k_i Y'_{\beta_m}(\lambda_{imp} r_i)$$
$$y_{i3} = -k_{i+1} J'_{\beta_m}(\lambda_{i+1,mp} r_i)$$
$$y_{i4} = -k_{i+1} Y'_{\beta_m}(\lambda_{i+1,mp} r_i)$$
$$C_{1\text{out}} = A_{\text{out}} J'_{\beta_m}(\lambda_{nmp} r_n) + B_{\text{out}} J_{\beta_m}(\lambda_{nmp} r_n)$$
$$C_{2\text{out}} = A_{\text{out}} Y'_{\beta_m}(\lambda_{nmp} r_n) + B_{\text{out}} Y_{\beta_m}(\lambda_{nmp} r_n)$$

and prime (′) denotes differentiation.

In the above matrix equation, λ_{imp} $(i \neq 1)$ may be written in terms of λ_{1mp} using Eq. (34). Subsequently, transverse eigencondition can be obtained by setting the determinant of the $(2n \times 2n)$ coefficient matrix in Eq. (44) equal to zero. Roots of which, in turn, yield the infinite number of eigenvalues λ_{1mp} corresponding to the first layer for each integer value of m. (Note that this step to find the eigenvalues can be reduced to setting the determinant of an $(n \times n)$—instead of $(2n \times 2n)$—matrix equal to zero. This can be done by applying the continuity of heat flux at the interfaces to eliminate one of the constants in Eq. (36).)

4.5. Determination of coefficients a_{imp} and b_{imp}

Coefficients a_{imp} and b_{imp} in the radial eigenfunction $R_{imp}(\lambda_{imp} r)$ (Eq. (36)) may be obtained from the following recurrence relationship, obtained from the ith interface condition (see Eqs. (16), (17)), valid for $i \in [1, n-1]$,

$$\begin{pmatrix} a_{i+1,mp} \\ b_{i+1,mp} \end{pmatrix} = \begin{pmatrix} J_{\beta_m}(\lambda_{i+1,mp} r_i) & Y_{\beta_m}(\lambda_{i+1,mp} r_i) \\ k_{i+1} J'_{\beta_m}(\lambda_{i+1,mp} r_i) & k_{i+1} Y'_{\beta_m}(\lambda_{i+1,mp} r_i) \end{pmatrix}^{-1} \times \begin{pmatrix} J_{\beta_m}(\lambda_{imp} r_i) & Y_{\beta_m}(\lambda_{imp} r_i) \\ k_i J'_{\beta_m}(\lambda_{imp} r_i) & k_i Y'_{\beta_m}(\lambda_{imp} r_i) \end{pmatrix} \begin{pmatrix} a_{imp} \\ b_{imp} \end{pmatrix} \tag{45}$$

where $b_{1mp} = -\frac{C_{1\text{in}}}{C_{2\text{in}}} a_{1mp}$ and a_{1mp} is arbitrary.

Clearly, two sets of eigenfunctions obtained with different a_{1mp} are proportional to each other and are equally valid solutions of the radial eigenvalue problem. Moreover, after the introduction of D_{mp} in the general solution, there is no need to retain a_{1mp} as a separate constant. (The above discussion is in fact true for any eigenvalue problem.)

4.6. Determination of coefficient D_{mp}

Coefficient D_{mp} in Eq. (40) may be obtained by applying the initial condition (20) and then making use of the orthogonality conditions in the radial and angular directions, as follows:

$$D_{mp} = \frac{1}{N_{\theta m} N_{rmp}} \sum_{i=1}^{n} \frac{k_i}{\alpha_i} \int_0^{\phi} \int_{r_{i-1}}^{r_i} r R_{imp}(\lambda_{imp} r) \times \Theta_m(\beta_m \theta) \overline{T}_i(r, \theta, t = 0)\, \mathrm{d}r\, \mathrm{d}\theta \tag{46}$$

5. Solution to the inhomogeneous steady state problem

The inhomogeneous steady state problem is solved using eigenfunction expansion method. The steady state temperature distribution, governed by Eq. (21), may be written as a generalized Fourier series in terms of angular eigenfunctions,

$$T_{\mathrm{ss},i}(r, \theta) = \sum_{m=1}^{\infty} \widehat{T}_{im}(r) \Theta_m(\beta_m \theta) \qquad r_{i-1} \leqslant r \leqslant r_i,\ 1 \leqslant i \leqslant n \tag{47}$$

Substituting Eq. (47) in Eq. (21) leads to an ODE for $\widehat{T}_{im}(r)$,

$$\frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r}\left(r \frac{\mathrm{d}\widehat{T}_{im}(r)}{\mathrm{d}r} \right) - \frac{\beta_m^2}{r^2} + \frac{\hat{g}_{im}(r)}{k_i} = 0 \qquad r_{i-1} \leqslant r \leqslant r_i,\ 1 \leqslant i \leqslant n \tag{48}$$

where the source term $g_i(r, \theta)$ is expanded in a generalized Fourier series as:

$$g_i(r, \theta) = \sum_{m=1}^{\infty} \hat{g}_{im}(r) \Theta_m(\beta_m \theta) \qquad r_{i-1} \leqslant r \leqslant r_i,\ 1 \leqslant i \leqslant n \tag{49}$$

where

$$\hat{g}_{im}(r) = \frac{1}{N_{\theta m}} \int_0^{\phi} g_i(r, \theta) \Theta_m(\beta_m \theta)\, \mathrm{d}\theta \tag{50}$$

Similarly, C_{in} and C_{out} in Eqs. (22) and (23) may be expanded in a generalized Fourier series to yield boundary conditions for ODE given in Eq. (48). Interface conditions for $T_{\mathrm{ss},i}(r, \theta)$, given in Eqs. (26)–(29), are also valid for $\widehat{T}_{im}(r)$.

Solution for Eq. (48) may be written as:

$$\widehat{T}_{im}(r) = A_{\mathrm{ss},i} r^m + B_{\mathrm{ss},i} r^{-m} + f_p(r) \tag{51}$$

where $f_p(r)$ is particular integral that can be obtained by application of method of variation of parameters or method of undetermined coefficients. Constants $A_{\mathrm{ss},i}$ and $B_{\mathrm{ss},i}$ may be evaluated using boundary and interface conditions for $\widehat{T}_{im}(r)$. It should be noted that $B_{\mathrm{ss},1} = 0$ when $r_0 = 0$.

6. Illustrative example

A three-layer semi-circular annular region ($r_0 \leqslant r \leqslant r_3$, $0 \leqslant \theta \leqslant \pi$; see Fig. 2) is initially at a uniform zero temperature. For time $t > 0$, the end surfaces for each layer at angle

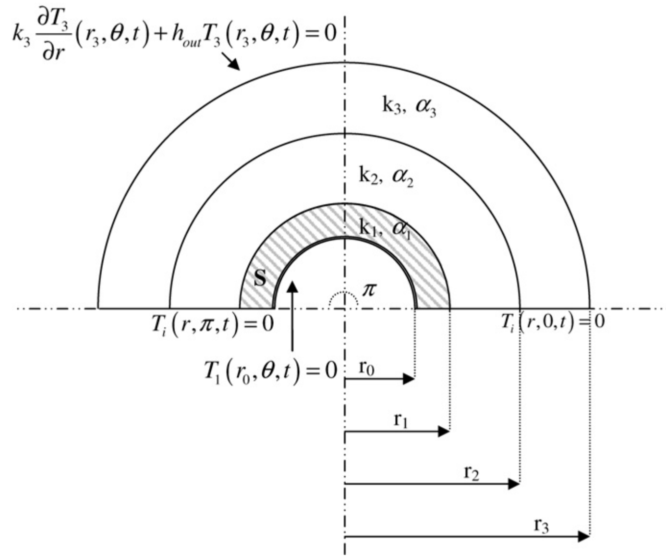


Fig. 2. Three layer semi-circular annular region example problem.

$\theta = 0$ and $\theta = \pi$ as well as inner radial surface at $r = r_0$ is maintained isothermal at zero temperature, while heat is convected into ambient, also at zero temperature, at the outer radial surface at $r = r_3$. These boundary conditions lead to $A_{\mathrm{in}} = 0$, $B_{\mathrm{in}} = 1$, $A_{\mathrm{out}} = k_3$ and $B_{\mathrm{out}} = h_{\mathrm{out}}$. In addition, uniformly distributed heat source of magnitude S is turned on at $t = 0$ in the first (innermost) layer.

Parameter values used for this problem are,

$$k_2/k_1 = 2, \quad k_3/k_1 = 4; \quad \alpha_2/\alpha_1 = 4, \quad \alpha_3/\alpha_1 = 9$$
$$r_1/r_0 = 2, \quad r_2/r_0 = 4, \quad r_3/r_0 = 6, \quad Bi_{\mathrm{out}} \equiv h_{\mathrm{out}} r_0 / k_1 = 1$$

It should be noted that, in the results that follow, r, t, and $T_i(r, \theta, t)$ are in the units of r_0, r_0^2/α_1 and $S r_0^2/k_1$, respectively. Moreover, for the boundary conditions chosen for this problem, $\omega_1 = 1$, $\omega_2 = 0$ and $\beta_m = m$ (see Table 1).

Steady-state solution for this particular problem can easily be obtained as,

$$T_{\mathrm{ss},i}(r, \theta) = \sum_{m=1}^{\infty} \widehat{T}_{im}(r) \sin(m\theta), \qquad i = 1, 2, 3 \tag{52}$$

where

$$\widehat{T}_{1m}(r) = A_{\mathrm{ss},1} r^m + B_{\mathrm{ss},1} r^{-m} - \frac{2}{\pi} \left(\frac{1 - \cos(m\pi)}{m(4 - m^2)} \right) \frac{S r^2}{k_1} \tag{53}$$

$$\widehat{T}_{im}(r) = A_{\mathrm{ss},i} r^m + B_{\mathrm{ss},i} r^{-m}, \quad i \neq 1 \tag{54}$$

The constants $A_{\mathrm{ss},i}$ and $B_{\mathrm{ss},i}$ ($i = 1, 2$ and 3) in Eqs. (53) and (54) can be evaluated by applying the steady-state interface and boundary conditions, which results in the matrix equation (55) (see the top of the next page) where $c_s = \frac{2}{\pi}(\frac{1-\cos(m\pi)}{m(4-m^2)})\frac{S}{k_1}$.

As in Eq. (40), double series solution for $\overline{T}_i(r, \theta, t)$ with $\beta_m = m$, can be written as:

$$\overline{T}_i(r, \theta, t) = \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} D_{mp} \mathrm{e}^{-\alpha_1 \lambda_{1mp}^2 t} \big(a_{imp} J_m(\lambda_{imp} r) + b_{imp} Y_m(\lambda_{imp} r) \big) \sin(m\theta) \tag{56}$$

$$\begin{bmatrix} A_{ss,1} \\ B_{ss,1} \\ A_{ss,2} \\ B_{ss,2} \\ A_{ss,3} \\ B_{ss,3} \end{bmatrix} = \begin{bmatrix} (A_{in}m + B_{in}r_0)r_0^{m-1} & (-A_{in}m + B_{in}r_0)r_0^{-m-1} & 0 & 0 & 0 & 0 \\ r_1^m & r_1^{-m} & -r_1^m & -r_1^{-m} & 0 & 0 \\ 0 & 0 & r_2^m & r_2^{-m} & -r_2^m & -r_2^{-m} \\ k_1 m r_1^{m-1} & -k_1 m r_1^{-m-1} & -k_2 m r_1^{m-1} & k_2 m r_1^{-m-1} & 0 & 0 \\ 0 & 0 & k_2 m r_2^{m-1} & -k_2 m r_2^{-m-1} & -k_3 m r_2^{m-1} & k_3 m r_2^{-m-1} \\ 0 & 0 & 0 & 0 & (A_{out}m + B_{out}r_3)r_3^{m-1} & (-A_{out}m + B_{out}r_3)r_3^{-m-1} \end{bmatrix}^{-1} \times \begin{bmatrix} (2A_{in} + B_{in}r_0)r_0 c_s \\ c_s r_1^2 \\ 0 \\ 2k_1 r_1 c_s \\ 0 \\ 0 \end{bmatrix} \tag{55}$$

Table 2
Transverse eigenvalues λ_{1mp} for the example problem

p	$m=1$	$m=3$	$m=5$	$m=7$	$m=9$	$m=11$	$m=13$	$m=15$	$m=17$	$m=19$
1	1.07454	2.08172	3.20819	4.33447	5.44768	6.54132	7.61439	8.67085	9.71599	10.7540
2	1.96189	2.72329	3.81788	4.99373	6.15741	7.27307	8.36040	9.45180	10.5476	11.6430
3	3.08567	3.57869	4.36917	5.29757	6.30213	7.37580	8.48674	9.59523	10.6948	11.7863
4	4.28626	4.71199	5.48113	6.47357	7.56946	8.68970	9.81260	10.9399	12.0723	13.2075
5	5.35901	5.68274	6.28854	7.11844	8.11952	9.23805	10.4118	11.5951	12.7682	13.9245
6	6.49831	6.78533	7.32467	8.06103	8.93373	9.88792	10.8843	11.9024	12.9351	13.9832
7	7.75062	7.99203	8.45839	9.12359	9.95917	10.9342	12.0077	13.1260	14.2440	15.3532
8	8.92835	9.12530	9.50822	10.0586	10.7550	11.5775	12.5102	13.5435	14.6627	15.8317
9	10.0234	10.2121	10.5813	11.1160	11.7965	12.6001	13.5009	14.4704	15.4808	16.5105
10	11.1955	11.3629	11.6919	12.1723	12.7913	13.5360	14.3950	15.3579	16.4107	17.5296

$$\begin{vmatrix} J_m(\lambda_{1mp}) & Y_m(\lambda_{1mp}) & 0 & 0 & 0 & 0 \\ J_m(2\lambda_{1mp}) & Y_m(2\lambda_{1mp}) & -J_m(\lambda_{1mp}) & -Y_m(\lambda_{1mp}) & 0 & 0 \\ y_{11} & y_{12} & y_{13} & y_{14} & 0 & 0 \\ 0 & 0 & J_m(2\lambda_{1mp}) & Y_m(2\lambda_{1mp}) & -J_m(\frac{4}{3}\lambda_{1mp}) & -Y_m(\frac{4}{3}\lambda_{1mp}) \\ 0 & 0 & y_{21} & y_{22} & y_{23} & y_{24} \\ 0 & 0 & 0 & 0 & C_{1out} & C_{2out} \end{vmatrix} = 0 \tag{57}$$

Application of the interface and boundary conditions to transverse eigenfunction yields the following eigencondition in a (6×6) determinant form (57) (see above), where

$$y_{11} = y_{21} = \frac{\lambda_{1mp}}{2}\big(J_{m-1}(2\lambda_{1mp}) - J_{m+1}(2\lambda_{1mp})\big)$$

$$y_{12} = y_{22} = \frac{\lambda_{1mp}}{2}\big(Y_{m-1}(2\lambda_{1mp}) - Y_{m+1}(2\lambda_{1mp})\big)$$

$$y_{13} = \frac{-\lambda_{1mp}}{2}\big(J_{m-1}(\lambda_{1mp}) - J_{m+1}(\lambda_{1mp})\big)$$

$$y_{14} = \frac{-\lambda_{1mp}}{2}\big(Y_{m-1}(\lambda_{1mp}) - Y_{m+1}(\lambda_{1mp})\big)$$

$$y_{23} = \frac{-2\lambda_{1mp}}{3}\left(J_{m-1}\left(\frac{4}{3}\lambda_{1mp}\right) - J_{m+1}\left(\frac{4}{3}\lambda_{1mp}\right)\right)$$

$$y_{24} = \frac{-2\lambda_{1mp}}{3}\left(Y_{m-1}\left(\frac{4}{3}\lambda_{1mp}\right) - Y_{m+1}\left(\frac{4}{3}\lambda_{1mp}\right)\right)$$

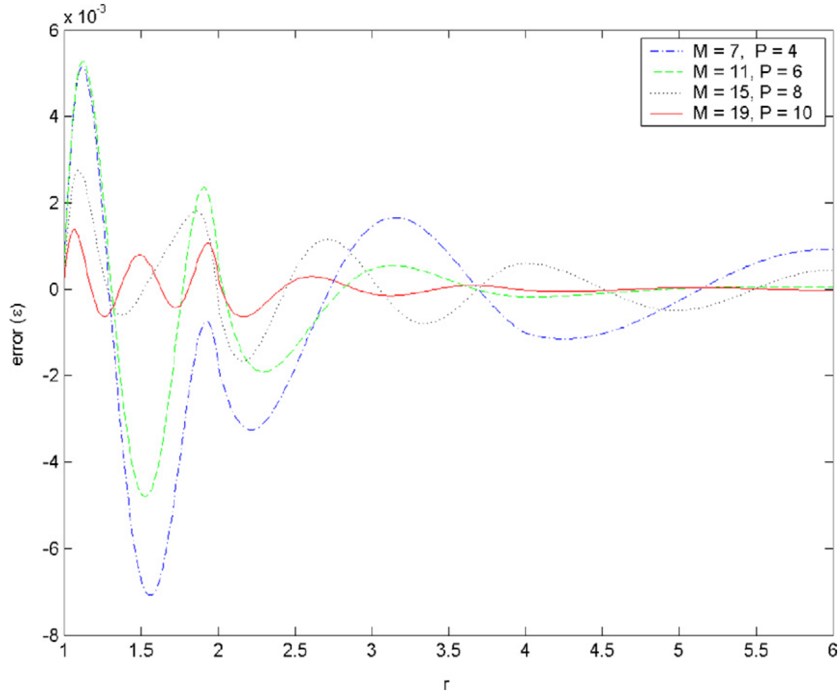
$$C_{1out} = J_m(2\lambda_{1mp}) + \frac{4}{3}y_{11}$$

$$C_{2out} = Y_m(2\lambda_{1mp}) + \frac{4}{3}y_{12}$$

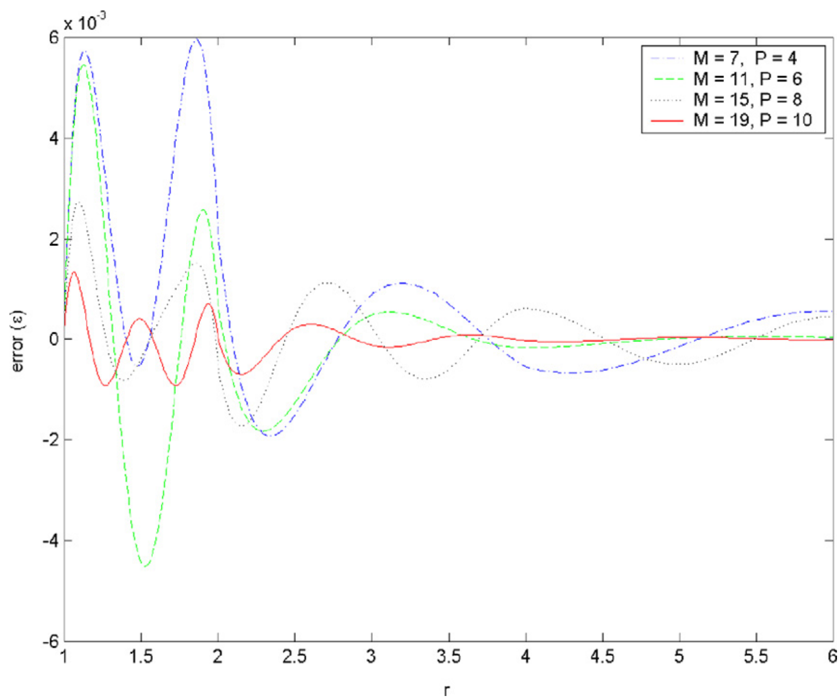
There exists infinite number of transverse eigenvalues (indexed by p) related to the first layer, λ_{1mp}, for each integer value of m. These eigenvalues λ_{1mp} are calculated by solving the above transcendental eigencondition with the help of Mathematica 5.1, a commercial mathematical package. Resulting eigenvalues for various values of m and p are shown in Table 2. Roots are searched in a user-defined window of size $\Delta\lambda$ using in-built functions. Successive eigenvalues are obtained by marching forward in the steps of $\Delta\lambda$ starting from zero. Since the roots are not distributed uniformly, the window size has to be kept very small in order not to miss any eigenvalue. Moreover, resulting eigenvalues are verified graphically to make sure that all eigenvalues within the interval were indeed captured. The above-mentioned scheme is not very efficient because a very small window size is required. Several methods have been developed so far to efficiently compute eigenvalues for 2-D Cartesian multi-layer problems [19,20]. Further research is necessary to develop an efficient and automated scheme for the current problem, which also guarantees that all eigenvalues are captured.

7. Results

For this particular problem, even integer values of m yield trivial values for D_{mp}. Therefore, transverse eigenvalues are obtained only for the odd integer values of m. The infinite series given in Eq. (56) is truncated at $p = P$ and $m = M$, leading to,



(a)



(b)

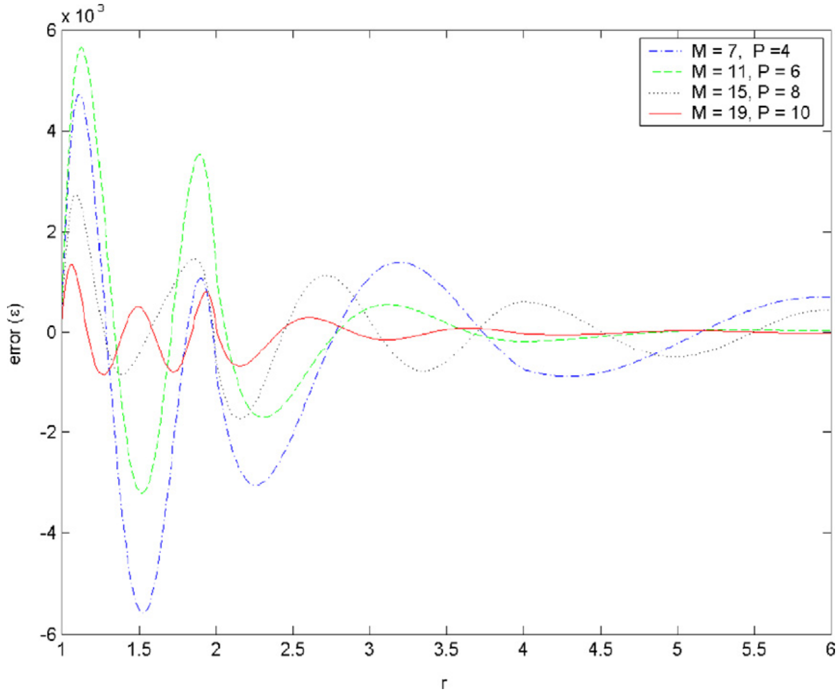
Fig. 3. Error in transient temperature distribution at $t = 0$ in radial direction at various angular positions: (a) $\theta = \pi/8$, (b) $\theta = \pi/4$, (c) $\theta = 3\pi/8$, (d) $\theta = \pi/2$.

$$\overline{T}_i(r,\theta,t) = \sum_{m=1}^{M}\sum_{p=1}^{P} D_{mp} e^{-\alpha_1 \lambda_{1mp}^2 t}\big(a_{imp} J_m(\lambda_{imp} r) + b_{imp} Y_m(\lambda_{imp} r)\big)\sin(m\theta) + \varepsilon_i(r,\theta,t,M,P) \tag{58}$$

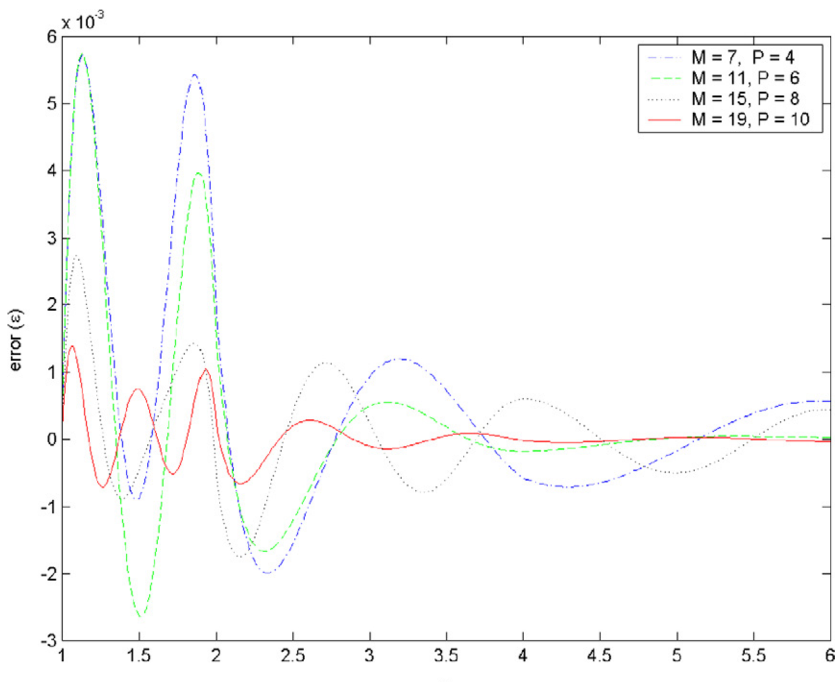
where $\varepsilon_i(r,\theta,t,M,P)$ is the truncation error.

Since λ_{1mp} increases with increasing m and p, it is obvious that for a given M and P, maximum truncation error occurs at $t = 0$. Moreover, since $\overline{T}_i(r,\theta,t=0) = -T_{ss,i}(r,\theta)$, therefore,

$$\varepsilon_i(r,\theta,t=0,M,P) = T_{ss,i}(r,\theta) + \sum_{m=1}^{M}\sum_{p=1}^{P} D_{mp}\big(a_{imp} J_m(\lambda_{imp} r) + b_{imp} Y_m(\lambda_{imp} r)\big)\sin(m\theta) \tag{59}$$



(c)



(d)

Fig. 3. *Continued.*

However, $T_{\mathrm{ss},i}(r,\theta)$ is also evaluated as a series solution, hence, the above equation can be written as:

$$\varepsilon_i(r,\theta,t=0,M,P)=\left(\sum_{m=1}^{M_{\mathrm{ss}}}\widehat{T}_{im}(r)\sin(m\theta)+\varepsilon_{\mathrm{ss},i}(r,\theta,M_{\mathrm{ss}})\right)+\sum_{m=1}^{M}\sum_{p=1}^{P}D_{mp}\big(a_{imp}J_m(\lambda_{imp}r)+b_{imp}Y_m(\lambda_{imp}r)\big)\sin(m\theta) \tag{60}$$

A good estimate of $\varepsilon_i(r,\theta,t=0,M,P)$ may be obtained only if $\varepsilon_{\mathrm{ss},i}(r,\theta,M_{\mathrm{ss}})\ll\varepsilon_i(r,\theta,t=0,M,P)$. The above requirement may be fulfilled by taking a large number of terms in the steady state series solution so as to minimize the steady state truncation error. Since the maximum difference between steady state temperatures obtained with $M_{\mathrm{ss}}=45$ and $M_{\mathrm{ss}}=50$ is of the order of 10^{-5}, therefore, series is truncated at $M_{\mathrm{ss}}=50$.

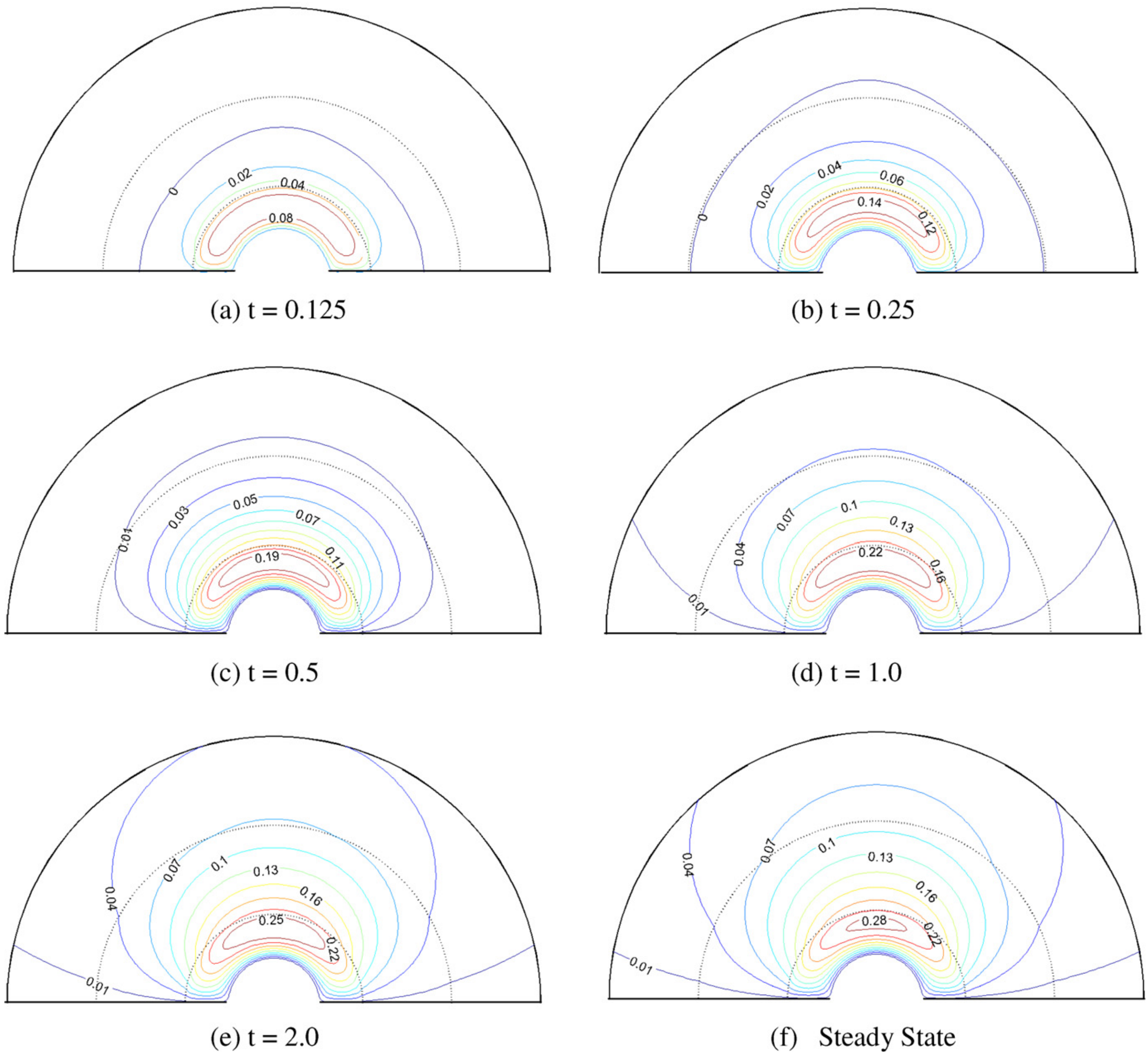


Fig. 4. Transient isotherms in three-layer annular region.

Plots of truncation error $\varepsilon_i(r, \theta, t = 0)$ for various values of M and P are shown in Fig. 3. Though plots are presented only for $\theta = \pi/8, \pi/4, 3\pi/8$ and $\pi/2$, it has been ensured that truncation error is of the same order for all values of θ. The $L-1$ % errors evaluated for the cases shown in Fig. 3, in the order of increasing M and P, are 1.33%, 0.84%, 0.63%, and 0.49%. Since the truncation error for $M = 19$ and $P = 10$ may be considered reasonably small, therefore, the series is truncated at these values of M and P.

Isotherms in the three-layer, semicircular, annular region are shown for different t values in Fig. 4. Additionally, angular and radial temperature variations are shown in Figs. 5 and 6, respectively. The steady-state solution is also shown for all the cases.

It may be noted that unsteady isotherms (in Fig. 4) and radial temporal variation curves (in Fig. 6) show jump in derivative at the layer interfaces due to step change in material properties. As heat source is turned on (at $t = 0$) in the first (innermost) layer, temperature grows rapidly with-in the first layer and then slowly decays in subsequent layers to satisfy convective boundary condition at the outside surface. Maximum temperature in the layered material is always found at $\theta = \pi/2$ and near the mid-section in the radial direction of the first layer.

8. Conclusions

In this paper, a closed form analytical solution to the two-dimensional, transient, heat conduction problem in polar coordinates, with multiple layers in the radial direction, is presented. Each layer can have spatially varying but time-independent volumetric heat source. Proposed solution is valid for any combination of homogenous boundary condition of the first or second kind in the angular direction. However, inhomogeneous bound-

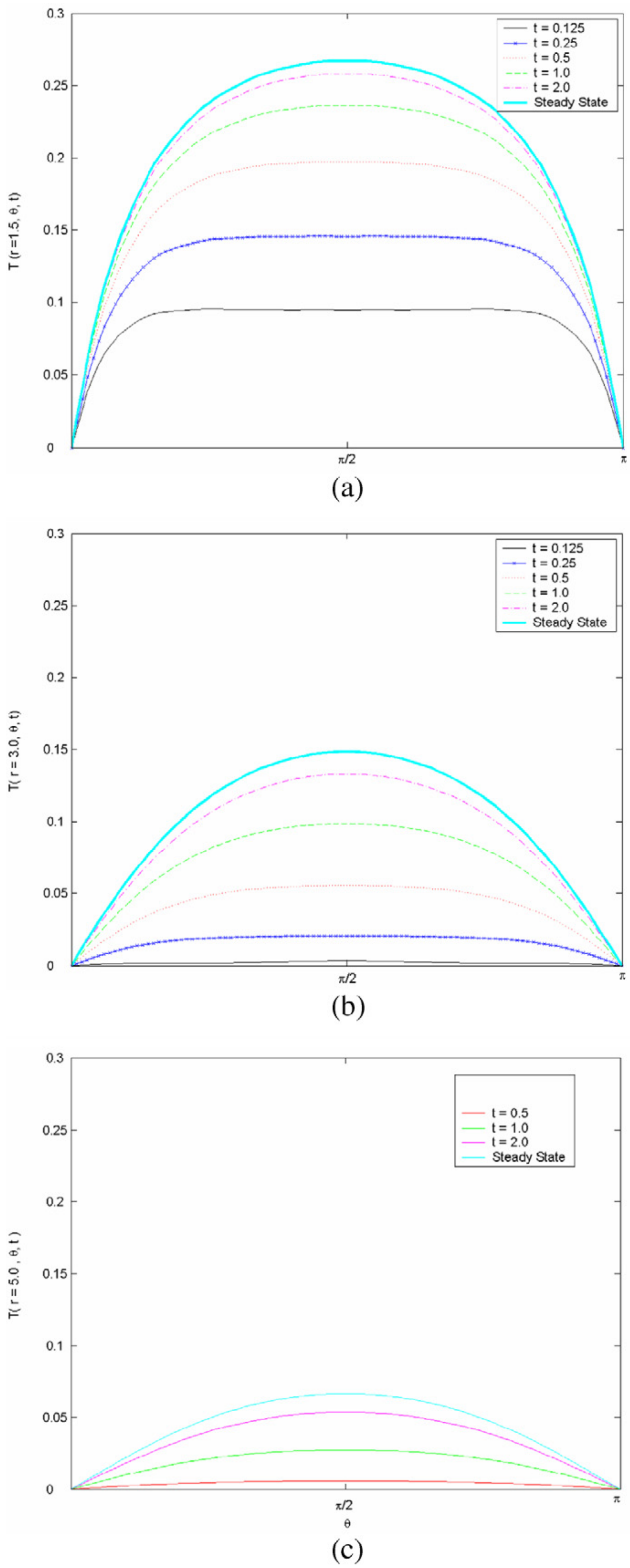


Fig. 5. Transient temperature distribution in angular direction at mid-sections of the three layers: (a) $r = 1.5$, (b) $r = 3.0$, (c) $r = 5.0$.

ary condition of the first, second or the third kind can be applied in the radial direction. Proposed solution is also applicable to the layered-structures with $r_0 = 0$.

It is noted that solution of multi-layer, two-dimensional heat conduction problem in polar coordinates is not analogous to the corresponding problem in multi-dimensional Cartesian coordinates (or 2-D cylindrical r–z coordinates). In the polar coordinates, dependence of the eigenvalues in the transverse direction on those in the other direction is not explicit. Absence of explicit dependence leads to a complete solution which does not have imaginary transverse eigenvalues. Numerical evaluation of the double series solution shows that a reasonable number of terms are sufficient to obtain results with acceptable errors for engineering applications.

Appendix A

A.1. Proof of the orthogonality condition

Let R_{imp} and R_{imq} be transverse eigenfunctions satisfying Eq. (31), thus

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R_{imp}}{\mathrm{d}r}\right) + \left(-\frac{\beta_m^2}{r^2} + \lambda_{imp}^2\right)R_{imp} = 0 \tag{A.1}$$

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R_{imq}}{\mathrm{d}r}\right) + \left(-\frac{\beta_m^2}{r^2} + \lambda_{imq}^2\right)R_{imq} = 0 \tag{A.2}$$

Boundary and interface conditions for $\overline{T}_i(r, \theta, t)$ (Eqs. (12)–(19)) are also valid for transverse eigenfunctions.

Since $\alpha_i \lambda_{imp}^2 = \alpha_1 \lambda_{1mp}^2$ (from Eq. (34)), we can write

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R_{imp}}{\mathrm{d}r}\right) + \left(-\frac{\beta_m^2}{r^2} + \frac{\alpha_1\lambda_{1mp}^2}{\alpha_i}\right)R_{imp} = 0 \tag{A.3}$$

Similarly,

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R_{imq}}{\mathrm{d}r}\right) + \left(-\frac{\beta_m^2}{r^2} + \frac{\alpha_1\lambda_{1mq}^2}{\alpha_i}\right)R_{imq} = 0 \tag{A.4}$$

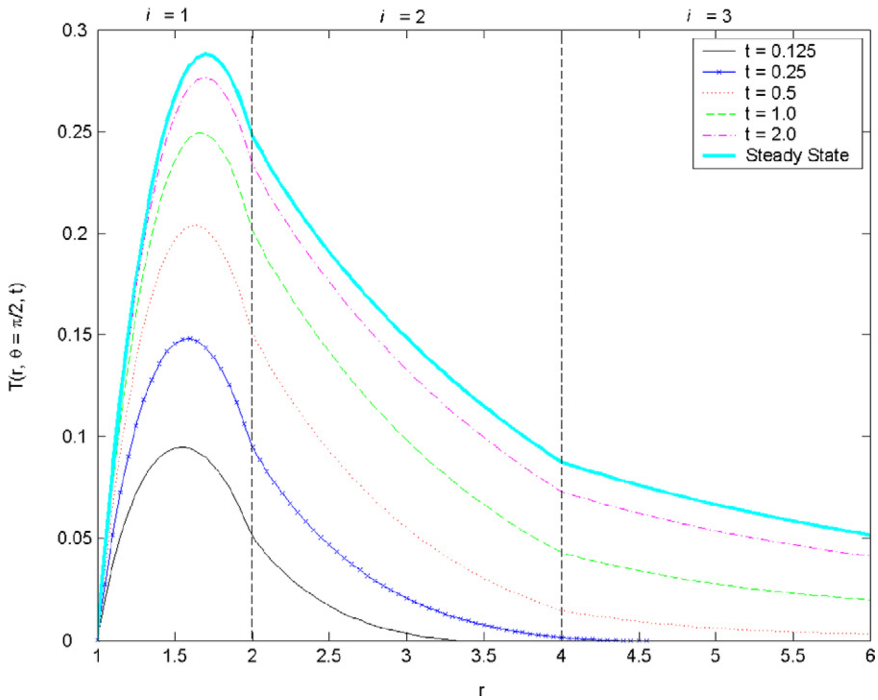
Multiplying (A.3) by R_{imq} and (A.4) by R_{imp} and subtracting, we get

$$R_{imq}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R_{imp}}{\mathrm{d}r}\right) - R_{imp}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R_{imq}}{\mathrm{d}r}\right) + \alpha_1\left(\frac{\lambda_{1mp}^2}{\alpha_i} - \frac{\lambda_{1mq}^2}{\alpha_i}\right)R_{imp}R_{imq} = 0 \tag{A.5}$$

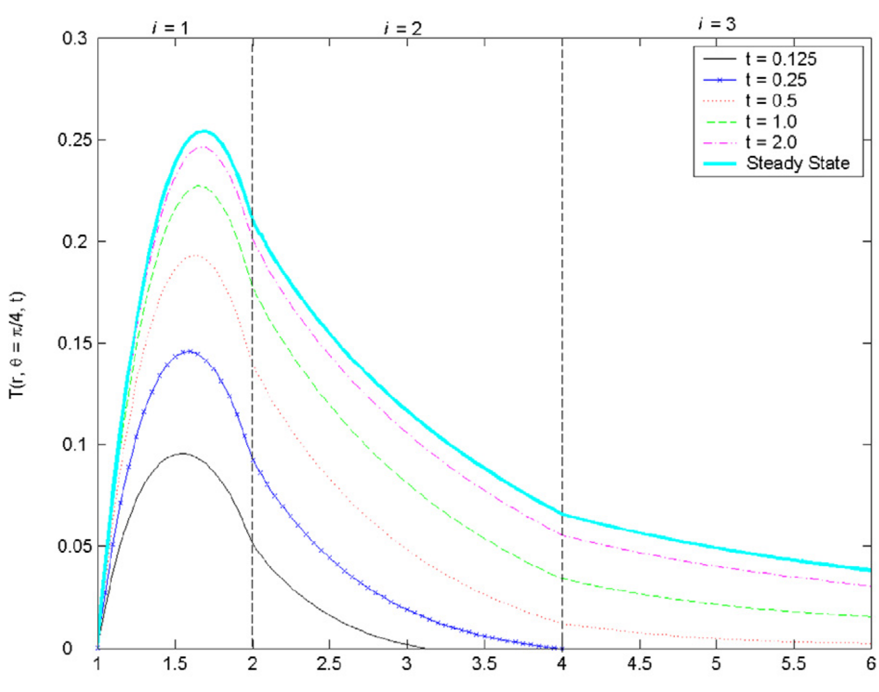
Now, operating with $\int_{r_{i-1}}^{r_i} r\,\mathrm{d}r$

$$\int_{r_{i-1}}^{r_i}\left(R_{imq}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R_{imp}}{\mathrm{d}r}\right)\right)\mathrm{d}r - \int_{r_{i-1}}^{r_i}\left(R_{imp}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R_{imq}}{\mathrm{d}r}\right)\right)\mathrm{d}r + \alpha_1\int_{r_{i-1}}^{r_i}\left(\frac{\lambda_{1mp}^2}{\alpha_i} - \frac{\lambda_{1mq}^2}{\alpha_i}\right)r R_{imp}R_{imq}\,\mathrm{d}r = 0 \tag{A.6}$$

Applying *integration by parts* twice on the first integral in the above equation,



(a)



(b)

Fig. 6. Transient temperature distribution in radial direction at $\theta = \pi/2$ and $\theta = \pi/4$.

$$\int_{r_{i-1}}^{r_i} \left(R_{imq} \frac{\mathrm{d}}{\mathrm{d}r}\left(r \frac{\mathrm{d}R_{imp}}{\mathrm{d}r} \right) \right) \mathrm{d}r = \left[r R_{imq} \frac{\mathrm{d}R_{imp}}{\mathrm{d}r} - r R_{imp} \frac{\mathrm{d}R_{imq}}{\mathrm{d}r} \right]_{r=r_{i-1}}^{r=r_i} + \int_{r_{i-1}}^{r_i} \left(R_{imp} \frac{\mathrm{d}}{\mathrm{d}r}\left(r \frac{\mathrm{d}R_{imq}}{\mathrm{d}r} \right) \right) \mathrm{d}r \tag{A.7}$$

Substituting Eq. (A.7) in Eq. (A.6) gives

$$\left[r R_{imq} \frac{\mathrm{d}R_{imp}}{\mathrm{d}r} - r R_{imp} \frac{\mathrm{d}R_{imq}}{\mathrm{d}r} \right]_{r=r_{i-1}}^{r=r_i} + \alpha_1 \int_{r_{i-1}}^{r_i} \left(\frac{\lambda_{1mp}^2}{\alpha_i} - \frac{\lambda_{1mq}^2}{\alpha_i} \right) r R_{imp} R_{imq} \, \mathrm{d}r = 0 \tag{A.8}$$

Multiplying the above equation by k_i and then summing over all i, we get

$$\sum_{i=1}^{n}\left[k_i r R_{imq}\frac{\mathrm{d}R_{imp}}{\mathrm{d}r} - k_i r R_{imp}\frac{\mathrm{d}R_{imq}}{\mathrm{d}r}\right]_{r=r_{i-1}}^{r=r_i} + \sum_{i=1}^{n}\frac{\alpha_1 k_i}{\alpha_i}\int_{r_{i-1}}^{r_i}\left(\lambda_{1mp}^2-\lambda_{1mq}^2\right) r R_{imp}R_{imq}\,\mathrm{d}r = 0 \tag{A.9}$$

Applying interface conditions (Eqs. (16)–(19)),

$$\left[k_n r R_{nmq}\frac{\mathrm{d}R_{nmp}}{\mathrm{d}r} - k_n r R_{nmp}\frac{\mathrm{d}R_{nmq}}{\mathrm{d}r}\right]_{r=r_n} - \left[k_1 r R_{1mq}\frac{\mathrm{d}R_{1mp}}{\mathrm{d}r} - k_1 r R_{1mp}\frac{\mathrm{d}R_{1mq}}{\mathrm{d}r}\right]_{r=r_0} + \sum_{i=1}^{n}\frac{\alpha_1 k_i}{\alpha_i}\int_{r_{i-1}}^{r_i}\left(\lambda_{1mp}^2-\lambda_{1mq}^2\right) r R_{imp}R_{imq}\,\mathrm{d}r = 0 \tag{A.10}$$

Now, from outer layer boundary condition (Eq. (13)), we have,

$$\left[A_{\text{out}}\frac{\mathrm{d}R_{nmp}}{\mathrm{d}r} + B_{\text{out}}R_{nmp}\right]_{r=r_n} = 0 \tag{A.11}$$

Similarly,

$$\left[A_{\text{out}}\frac{\mathrm{d}R_{nmq}}{\mathrm{d}r} + B_{\text{out}}R_{nmq}\right]_{r=r_n} = 0 \tag{A.12}$$

Multiplying (A.11) by $r_n R_{nmq}(r = r_n)$ and (A.12) by $r_n R_{nmp}(r = r_n)$ and subtracting,

$$A_{\text{out}}\left[k_n r R_{nmq}\frac{\mathrm{d}R_{nmp}}{\mathrm{d}r} - k_n r R_{nmp}\frac{\mathrm{d}R_{nmq}}{\mathrm{d}r}\right]_{r=r_n} = 0 \tag{A.13}$$

Now, we consider three different cases: (a) $A_{\text{out}} \neq 0$ and $B_{\text{out}} \neq 0$, (b) $A_{\text{out}} \neq 0$ and $B_{\text{out}} = 0$, (c) $A_{\text{out}} = 0$ and $B_{\text{out}} \neq 0$.

For cases (a) and (b), Eq. (A.13) reduces to

$$\left[k_n r R_{nmq}\frac{\mathrm{d}R_{nmp}}{\mathrm{d}r} - k_n r R_{nmp}\frac{\mathrm{d}R_{nmq}}{\mathrm{d}r}\right]_{r=r_n} = 0 \tag{A.14}$$

For case (c), Eqs. (A.11) and (A.12) imply that $R_{nmp}(r = r_n) = 0$ and $R_{nmq}(r = r_n) = 0$, respectively. Hence, Eq. (A.14) is also true for case (c).

Similarly, it can be shown that

$$\left[k_1 r R_{1mq}\frac{\mathrm{d}R_{1mp}}{\mathrm{d}r} - k_1 r R_{1mp}\frac{\mathrm{d}R_{1mq}}{\mathrm{d}r}\right]_{r=r_0} = 0 \tag{A.15}$$

Thus, in view of Eqs. (A.14) and (A.15), Eq. (A.10) yields

$$\left(\lambda_{1mp}^2-\lambda_{1mq}^2\right)\sum_{i=1}^{n}\frac{\alpha_1 k_i}{\alpha_i}\int_{r_{i-1}}^{r_i} r R_{imp}R_{imq}\,\mathrm{d}r = 0 \tag{A.16}$$

Since, for $p \neq q$, $\lambda_{1mp}^2-\lambda_{1mq}^2 \neq 0$ therefore

$$\sum_{i=1}^{n}\frac{k_i}{\alpha_i}\int_{r_{i-1}}^{r_i} r R_{imp}R_{imq}\,\mathrm{d}r = 0 \tag{A.17}$$

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